Quantum statistics via perturbation effects of preparation procedures

Andrei Khrennikov

International Center for Mathematical Modeling
in Physics and Cognitive sciences
MSI, Växjö University, S-35195, Sweden

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Abstract

We study the following problem: Is it possible to explain the quantum interference of probabilities in the purely corpuscular model for elementary particles? We demonstrate that (by taking into account perturbation effects of measurement and preparation procedures) we can obtain $\cos\theta$-perturbation (interference term) in probabilistic rule connecting preparation procedures for purely corpuscular objects. On one hand, our investigation demonstrated that there is nothing special in so called 'quantum probabilities': the right choice of statistical ensembles gives the possibility to escape all 'pathologies'. On the other hand, we found that the standard trigonometric interference of alternatives (observed, in particular, in quantum mechanics) is not the unique possibility to extend (disturb) the conventional probabilistic rule for addition of alternatives. There exist two other probabilistic rules that connect three preparation procedures: hyperbolic and hyper-trigonometric interferences.

1 Introduction

It is well known that the quantum probabilistic rule for interference of alternatives differs from the conventional probabilistic rule. For 'conventional probabilities', we have

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2, \tag{1}$$

for 'quantum probabilities', we have:

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + 2\sqrt{\mathbf{P}_1 \mathbf{P}_2} \cos \theta. \tag{2}$$

This difference in statistics is still one of the mysteries of modern physics. There are various explanations of the appearance of the $\cos \theta$-factor in quantum statistics. The common viewpoint is that new probabilistic behaviour which is observed in experiments with elementary particles is a consequence of the wave feature of these physical systems; see, for example, [1]-[18]. It is also commonly assumed that it would be impossible to explain the appearance of the $\cos \theta$-factor by using the purely corpuscular model for elementary particles. In fact, probabilistic transformation (2) was the main reason to introduce the principle of complementarity, Bohr [2].

In this paper we demonstrate that, despite the common opinion, probabilistic rule (2) can be derived in the purely corpuscular model as a consequence of the perturbation effects of preparation procedures. In particular, by using our method it is possible to simulate the interference of alternatives for macro-systems (for example, a kind of two-slit experiment for macro-balls). Our investigation was strongly motivated by the first chapters of Dirac's book [1]. There P. Dirac investigated the roots of quantum behaviour. He paid the large attention to the role of perturbations induced by preparation and measurement procedures. He rightly pointed out that the magnitude of these perturbations plays the crucial role in the transition from the classical theory to the quantum theory. However, he did not pay attention to the fact that these perturbations could produce quantum probabilistic behaviour, (2), in the purely corpuscular model. Therefore he, as many others, must use the wave particle duality (in particular, split of photons in the two slit experiment) to explain the origin of $\cos \theta$-factor in the quantum probabilistic rule (2): 'If the two components are now made to interfere, we should require a photon in one component to be able to interfere with one in the other', [1].

We also pay the main attention to perturbation effects of preparation procedures. We show that statistical deviations of sufficiently high magnitude induce the $\cos \theta$-factor for probability distributions of purely corpuscular objects.[1] An unexpected consequence of our analysis of possible probabilistic transformations (induced by preparation procedures) is that, beside trigonometric interference of alternatives, (2), we may obtain hyperbolic interference:

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 \pm 2\sqrt{\mathbf{P}_1 \mathbf{P}_2} \cosh \theta . \tag{3}$$

This interference can also be simulated.

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[1]Statistical deviations of negligibly small magnitude induce the conventional probabilistic rule.

Of course, we do not claim that our investigation implies that elementary particles are purely corpuscular objects. It may be that they have wave features and the principle of complementarity represents the important aspect of quantum reality. However, we definitely proved that quantum probabilistic rule (2) does not imply that we must use this principle to describe elementary particles (compare with Dirac [1], Bohr [2], Schrödinger [4], [5], Feynman [7]; compare also with [13]-[18]).

Through this paper we use the realist model (see [9], [10], [13] for the details) by that physical systems have objective properties. Values of physical observables can be considered as properties of the object. Of course, we do not claim that such a model can be used to describe quantum particles. However, we demonstrated that nontrivial 'quantum interference' of alternatives can be obtained even in the realist model.

We remark that (due to taking into account the perturbation effects of preparation procedures) our realist objective model does not differ essentially from so called contextualist model, see, for example, [10], [13]. In the latter model values of physical observables are merely determined by the context of an experiment. Perturbation effects of measurement procedures that play the crucial role in our investigation can be considered as the description of the experimental arrangement (context of an experiment). In principle, we can easily rewrite this paper in the purely contextualist framework. Here we should use sub-quantum model with hidden variables, see [13]. Instead of perturbation effects for physical variables, we can consider perturbation effects for hidden variables. In such a way we can reproduce all results of this paper.

In the objective realist framework perturbation effects are merely associated with preparation procedures (state=ensemble preparations). Preparation procedures perturb some (objective) properties of physical systems. By measurement we find numerical values describing these properties. In the contextualist framework perturbation effects are merely associated with measurement procedures. We cannot consider physical observables as numerical representations of properties of the object. The observed numerical values are created in the process of interaction between physical systems and measurement devices. These values depend not only on physical systems, but on the whole experimental arrangement. Nevertheless, such an interpretation of results of measurements does not exclude the possibility to provide a sub-quantum deterministic description, hidden variables (HV) model that reproduces predictions of quantum theory, see [13] for the details. The probabilistic model presented in this paper can be considered as such a HV-model (contextualist HV-model).

As we have already remarked, our investigation could not be considered

as the crucial argument against the wave particle dualism. The same can be said about the choice between the realist objective and contextualist interpretations of quantum mechanics. We just demonstrated that 'quantum probabilistic interference' could be obtained even in the realist objective model for corpuscular systems.

In this paper we discuss perturbation effects of preparation and measurement procedures. We remark that we do not follow to W. Heisenberg [6]; we do not study perturbation effects for individual measurements. We discuss statistical (ensemble) deviations induced by perturbations.[2]

We do not consider the relation of this model to the EPR-Bell considerations, [19], [20]. There is the large diversity of opinions on the origin of experimental violations of Bell's inequality, see, for example, [19]-[26]. Suppose that these violations really imply the impossibility to use objective realist interpretation of results of observations. Then our analysis would imply that EPR-Bell considerations was really the new step in the development of quantum theory with implications that, in fact, could not be obtained on the basis of the original formalism (despite the common opinion). Really, practically all founders of quantum theory (for example, De Broglie [27], Schrödinger [4], Dirac [1], Bohr [2], von Neumann [3], Bohm [28]) were sure that interference of probabilistic alternatives (that was demonstrated in the two slit experiment) could not coexist with purely corpuscular model for elementary particles. Especially extreme viewpoint was presented in works of Schrödinger. Finally, he criticized even attempts to use 'classical notion of a particle' in quantum theory [5], see [29] for the detailed analysis [3]

On the other hand, De Broglie [27], Dirac [1], Bohm [2], Feynman [7] tried to combine wave and corpuscular features of elementary particles by using different models. De Broglie imagined elementary particle as a singularity in the wave. It seems that in this model 'the mother wave produces a child-particle' by varying the space-time distribution. Bohm considered the pilot wave and elementary particle as indivisible whole. Here the wave is not more the mother of a particle. Von Neumann, Dirac, Feynman and the majority of quantum community considered corpuscular objects having unusual physical 'properties.' The wave features of elementary particles were exhibited via

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[2]Such an approach implies the statistical viewpoint to Heisenberg uncertainty relation: the statistical dispersion principle, see Ballentine [14], [15] for the details.

[3]However, compare with: 'The compulsion to replace the simultaneous happenings as indicated directly by the theory, by alternatives, of which the theory is supposed to indicated the respective probabilities, arises from the conviction that **what we really observe are particles - that actual events always concern particles not waves.**' see citation in [30], p.376, of Shrödinger's notes for a seminar he was giving in Dublin in 1952; here bold shrift is given by me.

superposition of alternatives in quantum state. On the other hand, it seems that Einstein was the adherent of the corpuscular model with the objective realist interpretation of physical observables, see [30], [31] for the detailed analysis.

In the connection with the EPR-Bell considerations we remark that (despite the common opinion) experimental violations of Bell's inequality can peacefully coexist with local realist HV-model with the contextualist interpretation of physical observables, see [13] and [32].

Finally, we note that HV-models are typically not taken into account (especially by 'real physicists'), because it is not useful to construct a complicated HV-model just to reproduce the standard results of (more simple) quantum formalism. In particular, such an argument is one of the main motivations for the rejection the Bohmian mechanics. In the opposite to such 'repetitive' HV-models, our model not only give the possibility to reproduce the standard quantum interference of probabilistic alternatives, but also predict new (hyperbolic) interference.

2 Statistical deviations produced by perturbations

2.1 Conventional probabilistic rule

Let Ω be an ensemble of physical systems having two physical properties a and b. These properties supposed to be *objective* properties of a physical system. To simplify considerations, we suppose that these properties can be described by dichotomic variables $a = 0, 1$ and $b = 0, 1$.

This situation can be described by the conventional probability model (Kolmogorov axiomatics, [33]). Here $(\Omega, \mathcal{F}, \mathbf{P}_\Omega)$ is a probability space: $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbf{P}_\Omega$ is a σ-additive probabilistic measure on $\mathcal{F}; a, b : \Omega \to \{0, 1\}$ are random variables. In our considerations the right choice of ensembles of physical systems will play the crucial role. Therefore we prefer to use for all probabilities indexes of corresponding ensembles.[4]

We set
$$\Omega_i^b = \{\omega \in \Omega : b(\omega) = i\}, i = 0, 1.$$

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[4]In Kolmogorov's axiomatics Ω is an arbitrary (abstract) set. However, in physical modeling it is important to use the right realization of this set. In our investigation Ω will be an ensemble of physical systems (for example, elementary particles). We underline that typically in mathematical works Ω is realized as the space of all possible sequences of trials, see [18] for the details.

As usual, we define conditional probabilities (Bayes' formula):

$$\mathbf{P}_{\Omega_i^b}(O) = \mathbf{P}_\Omega(O/\Omega_i^b) = \frac{\mathbf{P}_\Omega(O \cap \Omega_i^b)}{\mathbf{P}_\Omega(\Omega_i^b)}, i = 0, 1, \ O \in \mathcal{F}.$$

We shall be interested in conditional probabilities:

$$p_{ij}^{a/b} = \mathbf{P}_\Omega(a = j/b = i).$$

As usual we have:

$$p_{ij}^{a/b} = p_{ij}^{ab}/p_i^b,$$

where $p_{ij}^{ab} = \mathbf{P}_\Omega(a = j, b = i)$ and $p_i^b = \mathbf{P}_\Omega(\Omega_i^b)$.

Additivity of $\mathbf{P}_\Omega$ and the definition of conditional probabilities imply the formula of total probability:

$$\mathbf{P}_\Omega(a = j) = \mathbf{P}_\Omega(b = 0)\mathbf{P}_{\Omega_0^b}(a = j) + \mathbf{P}_\Omega(b = 1)\mathbf{P}_{\Omega_1^b}(a = j),$$

or

$$p_j^a \equiv \mathbf{P}_\Omega(a = j) = p_0^b p_{0j}^{a/b} + p_1^b p_{1j}^{a/b}, j = 1, 2. \tag{4}$$

We remark that the formula of total probability can be used as the *prediction rule*: if we know probabilities for the variable b and conditional probabilities for the variable a, then we can predict probabilities for the a. We call this rule, the *conventional probabilistic rule.*

2.2   Quantum probabilistic rule

The quantum mechanical formalism does not reproduce the formula of total probability; we cannot use the conventional probabilistic rule to predict probabilities in experiments with elementary particles. The right hand side of (4) is perturbed by a $\cos\theta$-factor. Often such a difference in transformation laws is interpreted as an evidence that objective realism could not be used in quantum framework. It seems that there is something mysterious in the appearance $\cos\theta-$perturbation in 'quantum formula of total probability'! Our aim is to provide probabilistic explanation of the appearance of this $\cos\theta$-factor.

Let φ be a quantum state and let Ω be a statistical ensemble of quantum systems prepared for φ. Let a and b be represented by operators $\hat{a}$ and $\hat{b}$ and $\{\psi_i\}_{i=0,1}$ and $\{\varphi_i\}_{i=0,1}$ be orthonormal bases of eigenvectors for $\hat{a}$ and $\hat{b}$. We can expend the quantum state φ with respect to basis $\{\phi_i\}_{i=0,1}$

$$\varphi = \sqrt{p_0^b}\varphi_0 + c^{i\xi}\sqrt{p_1^b}\varphi_1 .$$

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As $(\varphi, \varphi) = 1$, we have $p_0^b + p_1^b = 1$. By Born's probabilistic interpretation, $p_i^b = \mathbf{P}_\varphi(b = i) \equiv \mathbf{P}_\Omega(b = i)$.

We can also expend each φ_i with respect to the basis $\{\psi_i\}_{i=0,1}$:

$$\varphi_0 = e^{i\gamma_1} \sqrt{q_{00}^{a/b}} \psi_0 + e^{i\gamma_2} \sqrt{q_{01}^{a/b}} \psi_1,$$

$$\varphi_1 = e^{i\gamma_3} \sqrt{q_{10}^{a/b}} \psi_0 + e^{i\gamma_4} \sqrt{q_{11}^{a/b}} \psi_1,$$

where $\xi, \gamma_1, \ldots, \gamma_4$ are phases. Here (for values $j = 0, 1$)
$q_{ij}^{a/b} = \mathbf{P}_{\varphi_i}(a = j)$ (for states $i = 0, 1$).

We remark that, as each φ_i is normalized, we have:

$$q_{00}^{a/b} + q_{01}^{a/b} = 1, \; q_{10}^{a/b} + q_{11}^{a/b} = 1. \tag{5}$$

This is the standard normalization conditions for probabilities of alternatives. However, probabilities for quantum states (statistical ensembles used in quantum experiments) satisfy to another normalization condition. Orthogonality of eigenvectors corresponding to physical observables implies:

$$q_{00}^{a/b} + q_{10}^{a/b} = 1, \; q_{01}^{a/b} + q_{11}^{a/b} = 1. \tag{6}$$

This is so called condition of double stochasticity, see [10] for the details.

The standard calculations in the linear space imply that the quantum probabilistic (prediction) rule has the form

$$p_0^a = p_0^b q_{00}^{a/b} + p_1^b q_{10}^{a/b} + 2 \sqrt{p_0^b p_1^b q_{00}^{a/b} q_{10}^{a/b}} \cos\theta , \tag{7}$$

$$p_1^a = p_0^b q_{01}^{a/b} + p_1^b q_{11}^{a/b} - 2 \sqrt{p_0^b p_1^b q_{01}^{a/b} q_{11}^{a/b}} \cos\theta . \tag{8}$$

It is cruicial for our further considerations that the probabilities $q_{ij}^{a/b}$ (corresponding to quantum states φ_i) in general are not equal to probabilities $p_{ij}^{a/b}$ with respect to subensembles $\Omega_i^b, i = 0, 1$, of the ensemble Ω for the quantum state φ. In fact, $q_{ij}^{a/b}$ is the probability that $a = j$ in the ensemble $\tilde{\Omega}_i^b$ prepared for the quantum state φ_i. Therefore there is nothing surprising that conventional probabilistic rule (4) differs from quantum probabilistic rule (7), (8): probabilities p_i^a, p_i^b with respect to the ensemble Ω need not be connected with the aid of probabilities $q_{ij}^{a/b}$ with respect to new ensembles $\tilde{\Omega}_i^b$ by the ordinary formula of total probability. We shall explain how the formula of total probability produces the quantum probabilistic rule as a consequence of difference between the probabilities $p_{ij}^{a/b}$ and $q_{ij}^{a/b}$.

2.3 Statistical deviations

In general, we have three preparation procedures $\mathcal{E}, \mathcal{E}_0, \mathcal{E}_1$ such that $\mathcal{E}$ produces an ensemble Ω of physical systems and $\mathcal{E}_0 = F_0$ and $\mathcal{E}_1 = F_1$ are filters with respect to some property $b(= 0, 1)$ that produce ensembles $\tilde{\Omega}_0^b$ and $\tilde{\Omega}_1^b$. The p_i^a, p_i^b, are the probability distributions of a and b for the ensemble Ω; the $q_{ij}^{a/b}$ are probability distributions of a for the ensembles $\Omega_i^b, i = 0, 1$.

Preparations of ensembles $\tilde{\Omega}_i^b, i = 0, 1$, can be realized with the aid of filters $F_i, i = 0, 1$, which select particles from Ω having the property $b = i, i = 0, 1$. These selections perturb physical systems. The original distribution of a in Ω is changed.

We do not restrict our considerations to quantum experiments. We consider arbitrary preparation procedures for macro as well as micro-systems. We start with some evident manipulation with the ordinary formula of total probability for the ensemble Ω:

$$p_j^a = p_0^b p_{0j}^{a/b} + p_1^b p_{1j}^{a/b} = p_0^b q_{0j}^{a/b} + p_1^b q_{1j}^{a/b} + \delta_j(a, b),$$

where

$$\delta_j(a, b) = p_0^b(p_{0j}^{a/b} - q_{0j}^{a/b}) + p_1^b(p_{1j}^{a/b} - q_{1j}^{a/b}) = 2\sqrt{p_0^b p_1^b q_{0j}^{a/b} q_{1j}^{a/b}} \lambda_j,$$

where

$$\lambda_j \equiv \lambda_j(a, b) = \frac{p_0^b(p_{0j}^{a/b} - q_{0j}^{a/b}) + p_1^b(p_{1j}^{a/b} - q_{1j}^{a/b})}{2\sqrt{p_0^b p_1^b q_{0j}^{a/b} q_{1j}^{a/b}}} \tag{9}$$

We note that if $p_{0j}^{a/b} = q_{0j}^{a/b}$ and $p_{1j}^{a/b} = q_{1j}^{a/b}$, then $\delta_j(a, b) = 0$. This is the conventional probabilistic rule that we have in classical physics and in quantum physics in the absence of interference.

If perturbations produced by preparations of ensembles $\tilde{\Omega}_i^b$ (via filtrations of Ω)[5] are such that $\delta_j \neq 0$, we obtain 'nonconventional probabilistic rules'.

The coefficients δ_i and λ_i are called *statistical deviations* and *normalized statistical deviations*, respectively. The magnitudes of normalized statistical deviations will play the crucial role in our further considerations.

First we remark that if (for $j = 0, 1$)

$$|\lambda_j| \leq 1 \tag{10}$$

and the matrix of probabilities $(q_{ij}^{a/b})$ is double stochastic, we obtain the quantum probabilistic transformation (7), (**??**) by choosing $\lambda_0 = \cos\theta$ and $\lambda_1 = -\cos\theta$.

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[5]In general we need two copies of Ω to prepare $\tilde{\Omega}_0^b$ and $\tilde{\Omega}_1^b$.

We now present the general classification of probabilistic rules in nature. The general transformation of probabilities for three ensembles $\Omega, \tilde{\Omega}_i^b, i = 0, 1,$ has the form:

$$p_j^a = p_0^b q_{0j}^{a/b} + p_1^b q_{1j}^{a/b} + 2\sqrt{p_0^b p_1^b q_{0j}^{a/b} q_{1j}^{a/b}} \lambda_j, \tag{11}$$

where $\lambda_j, j = 0, 1$, are given by (9).

Depending on the magnitudes of normalized statistical deviations, we can obtain:

1) the trigonometric probabilistic rule, $\lambda_j = \cos \theta_j, j = 0, 1$;
2) hyperbolic probabilistic rule, $\lambda_j = \pm \cosh \theta_j, j = 0, 1$;
3) hyper-trigonometric probabilistic rule, for example,
$\lambda_0 = \pm \cosh \theta_1, \lambda_1 = \cos \theta_2$.

Here 'phases' θ are just special parameterizations for normalized statistical deviations.

In particular, the trigonometric probabilistic rule contains the conventional probabilistic rule, $\lambda_j = 0$, and the quantum probabilistic rule (under the additional condition that $(q_{ij}^{a/b})$ is a double stochastic matrix). Here normalized statistical deviations are relatively small, see (10). In the hyperbolic case they are quite large, namely:

$$|\lambda_j| \geq 1 \tag{12}$$

for both $j = 0, 1$. Here the order of perturbations via filtrations is essentially larger than, in particular, in quantum experiments.

Example 1. Let $\mathcal{E}, \mathcal{E}_0, \mathcal{E}_1$ produce following perturbations of probabilities:

$$\Delta_{0j} \equiv (p_{0j}^{a/b} - q_{0j}^{a/b}) p_0^b = 2\xi_{0j} \sqrt{p_0^b p_1^b q_{0j}^{a/b} q_{1j}^{a/b}};$$

$$\Delta_{1j} = (p_{1j}^{a/b} - q_{1j}^{a/b}) p_1^b = 2\xi_{1j} \sqrt{p_0^b p_1^b q_{0j}^{a/b} q_{1j}^{a/b}}.$$

where ξ_{0j} and ξ_{1j} are some coefficients. These coefficients determine the corresponding transformation rule for probabilities. If $\xi_{0j} + \xi_{1j} = 0$, we obtain the conventional rule. If $\xi_{0j} + \xi_{1j} \neq 0$, we obtain nonconventional rules; in particular, in quantum theory we have $|\xi_{0j} + \xi_{1j}| \leq 1$.

In a mathematical model we can describe filters $F_i, i = 0, 1$, by two maps

$$g_0 : \Omega_0^b \to \Omega_0^b, g_1 : \Omega_1^b \to \Omega_1^b$$

The pair of maps g_0 and g_1 induces the map
$g : \Omega \to \Omega$.

In principle, we can consider a more general model in that filters F_i can also change property b (some systems with $b = 0$ can be transformed into systems with $b = 1$ and vice versa). In such a model we have to consider general transformations $g : \Omega \to \Omega$ (i.e., without the restriction $g(\Omega_i^b) = \Omega_i^b$). We plan to study this model in further papers.

We shall use the symbols $\tilde{\Omega}_0^b$ and $\tilde{\Omega}_1^b$ to denote ensembles $g(\Omega_0^b)$ and $g(\Omega_1^b)$. We remark that in our model (with $g(\Omega_j^b) = \Omega_i^b$) new ensembles coincide with original ensembles as collections of physical systems (for example, particles), but physical properties are changed by filtrations.

Probability $\mathbf{P}_{\Omega_i^b}$ on the ensemble Ω_i^b (conditional probability $\mathbf{P}_\Omega(\cdot/\Omega_i^b)$) is lifted to ensemble $\tilde{\Omega}_i^b$ with the aid of map g_i :

$$\mathbf{P}_{\tilde{\Omega}_i^b}(O) = \mathbf{P}_{\Omega_i^b}(g_i^{-1}(O)).$$

We remark that by definition of conditional probability this probability is equal to $\mathbf{P}_\Omega(g_i^{-1}(O) \bigcap \Omega_i^b)/\mathbf{P}_\Omega(\Omega_i^b)$. Moreover, as g_i maps Ω_i^b on itself, we get that this probability is equal to :

$\mathbf{P}_\Omega(g_i^{-1}(O))/\mathbf{P}_\Omega(\Omega_i^b)$.

We note that the probability $\mathbf{P}_{\tilde{\Omega}_i^b}$ is nothing than the g_i-image of the conditional probability $\mathbf{P}_{\Omega_i^b}$: $\mathbf{P}_{\tilde{\Omega}_i^b} = g_i^* \mathbf{P}_{\Omega_i^b}$.

It is assumed that ensembles $\tilde{\Omega}_i^b$ are equipped by some σ-algebras and maps g_i are measurable.

We note that in quantum theory probability distributions $\mathbf{P}_{\varphi_i}$ (for states φ_i) are nothing than $\mathbf{P}_{\tilde{\Omega}_i^b}, i = 0, 1$. [6]

In general we introduce probabilities

$$q_{ij}^{a/b} \equiv \mathbf{P}_{\tilde{\Omega}_i^b}(a = j) = \mathbf{P}_{\Omega_i^b}(a(g_i(\omega)) = j).$$

By definition of conditional probability

$$q_{ij}^{a/b} = \mathbf{P}_\Omega(a(g_i)(\omega)) = j/b(\omega) = i) = \frac{q_{ij}^{ab}}{p_i^b}, i, j = 0, 1,$$

where

$q_{ij}^{ab} = \mathbf{P}_\Omega(a(g_i)(\omega)) = j, b(\omega) = i)$.

In particular, in quantum theory $q_{ij}^{a/b} = \mathbf{P}_{\varphi_i}(a = j)$.

The coefficients $\lambda_j(a, b)$ corresponding to perturbations of probabilities by filtrations can be represented as

$$\lambda_j(a, b) = \frac{(p_{0j}^{ab} - q_{0j}^{ab}) + (p_{1j}^{ab} - q_{1j}^{ab})}{2\sqrt{q_{0j}^{ab} q_{1j}^{ab}}} \tag{13}$$

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[6] Later we shall present concrete maps g_i that reproduce quantum probability rule.

Remark. (Contextualism) In the contextualist HV-model preparation procedures $\mathcal{E}_0$ and $\mathcal{E}_1$ disturb not physical variable a, but hidden variable ω.

Remark. (Contextualism without HV) We can even forget about HV and consider ω as purely mathematical 'chance parameter.' In this framework all our considerations are based on the observation that measurements of the a-variable for three different statistical ensembles Ω, $\tilde{\Omega}_0^b$ and $\tilde{\Omega}_1^b$ are performed via three different measurement procedures. Therefore they are described by different random variables $a(\omega), \tilde{a}^{(0)}(\omega) = a(g_0(\omega)), \tilde{a}^{(1)}(\omega) = a(g_1(\omega))$. We ask ourself: What kind of transformations connecting probability distributions of these random variables can we obtain in different experiments? It seems that it is too general formulation of the problem (however, see remark at the end of the paper). It would be more natural to consider only transformations that are perturbations of the standard formula of total probability (based on the Bayes' rule for conditional probabilities). In fact, we proved that there are only three types of such probabilistic transformations: trigonometric, hyperbolic and hyper-trigonometric. The creation of quantum formalism was the important discovery in probability theory: it was found (on the basis of interference experiments) that in some experimental situations the formula of total probability has to be disturbed by trigonometric perturbation. However, the probabilistic roots of this perturbation term were not found. This implied the creation of the model of micro-reality based on the wave-particle duality. The possibility of non-trigonometric interferences was not observed.

3 Simulation of trigonometric and hyperbolic interference

Measurements producing trigonometric and hyperbolic interferences that are presented in this section can be easily simulated by using just a pseudo-random generator of numbers uniformly distributed on the segment $[0, 1]$.

Let $\Omega = [0, 1], \mathcal{F}$ - the σ-algebra of Borel sets, $\mathbf{P} = dx$ is the uniform probability distribution (Lebesque measure). Let

$$c(\omega) = \begin{cases} 1, x \in [0, \frac{1}{3}) \\ 0, x \in [\frac{1}{3}, 1] \end{cases}$$

$$a(\omega) = \begin{cases} 1, x \in (\frac{1}{4}, \frac{3}{4}] \\ 0, x \notin (\frac{1}{4}, \frac{3}{4}] \end{cases}$$

So we have ensembles $\Omega_1^b = [0, \frac{1}{3}), \Omega_0^b = [\frac{1}{3}, 1]$; and probabilities:

$p_1^b = \mathbf{P}_\Omega(\Omega_1^b) = \frac{1}{3}$, $p_0^b = \mathbf{P}_\Omega(\Omega_0^b) = \frac{2}{3}$;

$p_{01}^{ab} = \mathbf{P}_\Omega(a(\omega) = 1, b(\omega) = 0) = \frac{5}{12}, p_{00}^{ab} = \mathbf{P}_\Omega(a(\omega) = 0, b(\omega) = 0) = \frac{1}{4}$,

$p_{11}^{ab} = \mathbf{P}_\Omega(a(\omega) = 1, b(\omega) = 1) = \frac{1}{12}, p_{10}^{ab} = \mathbf{P}_\Omega(a(\omega) = 0, c(\omega) = 1) = \frac{1}{4}$.

Let us consider maps g_i which describe changes of a in the processes of filtrations:

$g_0 : \Omega_0^b \to \Omega_0^b$ such that $g_0([\frac{1}{3}, \beta]) = [\frac{1}{3}, \frac{3}{4}]$ and $g_0((\beta, 1]) = (\frac{3}{4}, 1]$.
So $\tilde\Omega_0^b = g_0(\Omega_0^b) = [\frac{1}{3}, 1]$.
$g_1 : \Omega_1^b \to \Omega_1^b$ such that $g_1([0, \alpha]) = [0, \frac{1}{4}]$ and $g_1((\alpha, \frac{1}{3})) = (\frac{1}{4}, \frac{1}{3})$.
So $\tilde\Omega_1^b = g_1(\Omega_1^b) = [0, \frac{1}{3})$.

Here α and β are parameters such that $0 < \alpha < \frac{1}{3}$ and $\frac{1}{3} < \beta < 1$.

Of course, there exist various maps g_0 and g_1 which satisfy the above conditions. Different maps correspond to different physical realizations of filters. We shall see that by varying parameters α and β we can obtain classical, trigonometric (in particular, quantum) and hyperbolic probabilistic behaviours. We have

$q_{01}^{ab} = \mathbf{P}_\Omega(a(g_0(\omega)) = 1, b(\omega) = 0) = \beta - \frac{1}{3}$;
in the same way $q_{00}^{ab} = 1 - \beta; q_{11}^{ab} = \frac{1}{3} - \alpha; q_{10}^{ab} = \alpha$.

To generate quantum behaviour, we have to have a double stochastic matrix of probabilities $(q_{ij}^{a/b})$.

We have: $q_{ij}^{a/b} = \mathbf{P}_{\tilde\Omega_i^b}(a = j)$. So
$q_{01}^{a/b} = \frac{3\beta - 1}{2}; q_{00}^{a/b} = \frac{3 - 3\beta}{2}; q_{11}^{a/b} = 1 - 3\alpha, q_{10}^{a/b} = 3\alpha$.
The matrix $(q_{ij}^{a/b})$ is always stochastic:
$q_{00}^{a/b} + q_{01}^{a/b} = \mathbf{P}_{\tilde\Omega_0^b}(a = 0) + \mathbf{P}_{\tilde\Omega_0^b}(a = 1) = 1(= \frac{3 - 3\beta}{2} + \frac{3\beta - 1}{2})$;
$q_{10}^{a/b} + q_{11}^{a/b} = \mathbf{P}_{\tilde\Omega_1^b}(a = 0) + \mathbf{P}_{\tilde\Omega_1^b}(a = 1) = 1(= 1 - 3\alpha + 3\alpha)$.
It is double stochastic if
$q_{00}^{a/b} + q_{10}^{a/b} = \frac{3 - 3\beta}{2} + 3\alpha = 1; q_{01}^{a/b} + q_{11}^{a/b} = \frac{3\beta - 1}{2} + 1 - 3\alpha = 1$.
Thus

$$\beta = \frac{1}{3} + 2\alpha \tag{14}$$

(we note that if α varies between 0 and 1/3, then β varies between 1/3 and 1).

Double stochasticity implies: $q_{01}^{a/b} = q_{10}^{a/b} = 3\alpha$ and $q_{00}^{a/b} = q_{11}^{a/b} = 1 - 3\alpha$. In general by using formula (13) we get

$$\lambda_0 = \frac{(\beta - \frac{3}{4}) + (\frac{1}{4} - \alpha)}{2\sqrt{\alpha(1 - \beta)}} = \frac{(\beta - \alpha - \frac{1}{2})}{2\sqrt{\alpha(1 - \beta)}};$$

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$$\lambda_1 = \frac{(\frac{1}{2} + \alpha - \beta)}{2\sqrt{(\beta - \frac{1}{3})(\frac{1}{3} - \alpha)}}.$$

If $\beta = \alpha + \frac{1}{2}$, we generate convential probabilistic rule in that $\lambda_0 = \lambda_1 = 0$.

In particular, we get this rule in the case of filtrations such that: $\alpha = 1/4$ and $\beta = 3/4$. In particular, such filtrations can be realized by identical maps: $g_0(x) = x, g_1(x) = x$. These filters are ideal: they do not perturb the a-variable at all. However, there are infinitely many filters g_0, g_1 with $\alpha = 1/4$ and $\beta = 3/4$ that produce essential perturbations for individual physical systems: the variable a can be strongly changed in some points. Nevertheless, these filters do not produce interference of probabilistic alternatives, since statistical deviations are negligibly small (compare with Ballentine's analysis of Heisenberg uncertainty principle, [15], see also [13]).

We want to obtain nontrivial quantum behaviour: $\lambda_0 = \cos\theta_0 = -\lambda_1 \neq 0$. First we use the condition of double stochasticity (14):

$$\lambda_0 = \frac{\alpha - \frac{1}{6}}{2\sqrt{\alpha(\frac{2}{3} - 2\alpha)}} = -\lambda_1.$$

We obtain trigonometric interference if

$$|\lambda_0| = \frac{|\alpha - \frac{1}{6}|}{2\sqrt{2\alpha(\frac{1}{3} - \alpha)}} \leq 1.$$

Direct computation demonstrates that $\lambda_0'(\alpha) > 0$ for $\alpha \in (0, 1/3)$. Thus the interference term $\lambda_0(\alpha)$ is the increasing function of the perturbation parameter α and $\lambda_0(0) = -\infty, \lambda_0(\frac{1}{3}) = +\infty, \lambda_0(\frac{1}{6}) = 0, \lambda_0(\alpha_\pm) = \pm 1$, where $\alpha_\pm = \frac{3 \pm 2\sqrt{2}}{18}$.

Therefore perturbations (due to filtration) varying in the segment $[\alpha_-, \alpha_+]$ generate quantum behaviour: $\lambda_0(\alpha) = \cos\theta_0$ is varying from -1 to 1. The phase-angle θ_0 can vary from $[\pi, 2\pi]$ (by a parameterization of the perturbation variable we can get θ_0 varying from 0 to π).

Remark. (Contextualist model with hidden variable) We can use the contextualist interpretation of quantum mechanics. Here α and β are parameters determining the context of an experiment, ω is a hidden variable.

We now consider the extreme values of $\alpha : \alpha = \alpha_\pm$ (maximal interference) and $\alpha = \frac{1}{6}$ (decoherence). Quantities $\pi_0 = \frac{1}{4} - \alpha$ and $\pi_1 = \frac{3}{4} - \beta$ can be used as measures of perturbation of a due to the transitions

$$\Omega_0^b \to \tilde{\Omega}_0^b \quad \text{and} \quad \Omega_1^b \to \tilde{\Omega}_1^b. \tag{15}$$

Let $\alpha = 1/6$. Here $\pi_0 = \frac{1}{12}; \pi_1 = \frac{1}{12}$. The absence of interference is characterized by the symmetric shift of the a-property under the transition (15).

We remark that there exists only one point α that satisfies to the decoherence condition: $\pi_0 = \pi_1$, namely $\alpha = 1/6$ (under condition $\beta = \frac{1}{3} + 2\alpha$).

We have to differ 'classical situation' from 'quantum decoherence'. In the first case $\delta(a, b) = 0$ as the result of the precise filtration: $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$. In the second case filtration induces essential perturbations. However, these perturbations compensate each other.

In fact, the condition of double stochasticity (see(14)) plays here the cruicial role: it is a kind of constraint between perturbations induced by preparations of ensembles $\tilde{\Omega}_0^b$ and $\tilde{\Omega}_1^b$. It seems that this condition is the root of 'quantum mystery'. On the other hand, $\cos\theta$−factor (interference) can be induced by 'classical preparation procedures' for macro-systems.

The quantity

$\xi(\alpha) = |\pi_0(\alpha) - \pi_1(\alpha)| = |\alpha - \frac{1}{6}|$

can be used as a measure of asymmetry of perturbations (15). In the case of decoherence $\xi = \xi(\frac{1}{6}) = 0$. For $\alpha \in [\alpha_-, \alpha_+], \xi(\alpha) = |\alpha - 1/6|$ yields the maximal value for $\alpha = \alpha_\pm$. Here $\xi = \xi_{\max} = \frac{\sqrt{2}}{9}$. Thus maximum of interference ($|\cos\theta_0| = 1$) corresponds to maximal asymmetry of perturbations.

What kind of behaviour would we observe for perturbations $\alpha \notin (\alpha_-, \alpha_+)$? We see that, instead of 'trigonometric quantum behaviour', we obtain 'hyperbolic quantum behaviour'.

If $\alpha > \alpha_+$ (but $\alpha < 1/3$), then $\lambda_0(\alpha)$ can be represented as $\cosh\theta_0$; if the perturbation parameter α is varying from $\alpha_+$to 1/3, then $\cosh\theta_0$ is varying from 1 to $+\infty$. So the ' hyperbolic phase' can be chosen belonging to $[0, +\infty)$. If $\alpha < \alpha_-$ (but $\alpha > 0$), then $\lambda_0(\alpha)$ can be represented as $-\cosh\theta_0$. The variation of α from α_- to 0, implies the variation λ_0 from -1 to $-\infty$. So $\theta_0 \in [0, \infty)$.

'Hyperbolic quantum behaviour' may be induced by filters having sufficiently strong perturbations. In our model there are the thresholds $\alpha = \alpha_\pm$. At these levels of perturbations 'quantum trigonometric behaviour' is transformed to 'quantum hyperbolic behaviour'. Our model gives the possibility to simulate such a transition by using macro-systems. It may be that such a transition can be observed for some 'natural' physical processes.

The formalism of hyperbolic quantum mechanics (representation of the probabilistic rule (3) in a linear space) was developed in [34]. Instead of complex numbers, we have to work with so called hyperbolic numbers, see [35], p. 21. The development of hyperbolic quantum mechanics can be interesting for comparative analysis with standard quantum mechanics. In particular, we

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clarify the role of complex numbers in quantum theory. Complex (as well as hyperbolic) numbers were used to linearize nonlinear probabilistic rule (that in general could not be linearized over real numbers). Another interesting feature of hyperbolic quantum mechanics is the violation of the principle of superposition. Here we have only some restricted variant of this principle.

We remark that the quantum probabilistic transformation
$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + 2\sqrt{\mathbf{P}_1\mathbf{P}_2}\cos\theta$$
gives the possibility to predict the probability $\mathbf{P}$ if we know probabilities $\mathbf{P}_1$ and $\mathbf{P}_2$. In principle, there might be created theories based on arbitrary transformations:
$$\mathbf{P} = F(\mathbf{P}_1, \mathbf{P}_2).$$

It may be that some rules have linear space representations over 'exotic number systems', for example, p-adic numbers [36].

Preliminary analysis of probabilistic foundations of quantum mechanics (that induced the present investigation) was performed in the books [18] and [36] (chapter 2); a part of results of this paper was presented in preprints [37],[38].

Acknowledgements

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